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Performance of an Aided Redundant Navigator in Normal and Failure Modes

J. V. Harrison,* M. B. Adams,† E. G. Gai,‡ K. C. Daly,‡ and S. D. Ginter§
The Charles Stark Draper Laboratory, Cambridge, Mass.

The performance of a redundant strapdown inertial navigation system in the normal mode, failure mode, and reconfiguration mode is characterized by a single figure-of-merit representing the probability of mission success. A sensitivity matrix derived from a linearized error analysis relates attitude, position, and velocity errors to initial condition errors as well as inertial sensor static and dynamic errors. The implementation of state vector updates is described. Numerical results are presented for an orbit insertion mission.

Introduction

LINEARIZED error analyses have frequently been used to obtain a statistical characterization of navigation system performance.[1,2] This paper presents a rigorous extension of these techniques to the analysis of the performance of an aided redundant strapdown inertial navigation system.

The formulation of this analysis in terms of the sensitivity of state errors to error sources provides considerable insight into the performance of the redundant system in the normal mode (i.e., in the absence of failures), as well as in any of the operational modes resulting from failures or from failure detection and identification (FDI) decisions which affect the sensor configuration. The design of redundant aided inertial navigation systems affords numerous opportunities to tradeoff system elements in an effort to satisfy performance objectives under various design constraints. The use of the linearized analysis described in this paper in combination with an analysis of FDI performance[3] provides the quantitative results which make intelligent tradeoffs possible. The formulation of the analysis is sparing of computer resources and inexpensive to use once it has been implemented.

The elements of the sensitivity matrices represent the sensitivity of system performance to each of the individual error sources or sensor failures which are modeled as deterministic constants. Using these sensitivity matrices, the statistics of the state errors (or their estimates) can be determined from the statistics of the error sources (or their estimates) at any time.

The effects of error sources which are properly modeled as random processes can be evaluated by reformulating the analysis as a conventional covariance analysis[4] or by computing their effects separately and adding them to the covariance of the state errors obtained from the sensitivity analysis. In the authors' experiences,[5] the random errors which are characteristic of navigation quality inertial sensors seldom contribute significantly to overall system performance when all other error sources are considered.

Navaid updates are formulated in terms of their effect on the statistics of the error sources rather than in terms of their effect on the sensitivities. This results in a computationally efficient evaluation of the extent to which navigation system performance can be improved, and the effects of system failures mitigated, through the use of navaids.

The failure sensitivities, which are computed in the linearized analysis, establish the levels of failures which significantly degrade system performance. These levels represent design requirements for the FDI system. Once the FDI system is designed, the analysis of navigation system performance can be combined with an analysis of FDI performance to choose FDI thresholds which maximize the probability of success of the system.[6]

The remainder of the paper is divided into two major sections. The first section presents the analysis. The recursive equations used to propagate the sensitivity matrices along a nominal trajectory are developed, and the equations used to update the statistics of the state errors are derived. The modifications of these results to account for system failures are then discussed.

The second section presents results for an aided redundant strapdown inertial navigation system consisting of five gyros and five accelerometers in a conical array. Star-sensor and ranging measurements are included to indicate the effects of navaid updates on system performance in normal and failure modes.

Analysis

The linearized error analysis of the redundant navigation system is performed via a sensitivity matrix which relates the attitude, position, and velocity errors to navigation system error sources. In this section the linearized relations are developed for four operational cases. These cases are comprised of the four combinations of the presence or absence of a system failure and the availability or inavailability of information from navigation aids.

Define the state vector $x(t)$ as

$$x^T(t) = [q^T(t), r^T(t), v^T(t)] \tag{1}$$

consisting of the vehicle attitude quaternion $q(t)$, position $r(t)$, and velocity $v(t)$. Let the vector e represent the navigation system error sources, i.e.,

$$e^T = [\delta x^T(t_0), \delta b^T, \delta c^T, \delta\gamma^T] \tag{2}$$

where $\delta x(t_0)$ is the error in the initial attitude, position, and velocity; δb is the error in the inertial sensor constant biases; δc is the error in coefficients of inertial sensor scale factors, misalignments, and g- and g^2-sensitive terms; and $\delta\gamma$ is the bias error associated with navigation aid(s). Here δb, δc, and $\delta\gamma$ are modeled as random constants so that e is a time-invariant vector.

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*Member of the Technical Staff.
†Student, Dept. of Aero. and Astro. MIT, Draper Fellow. Member AIAA.
‡Member of the Technical Staff. Member AIAA.
§Student, Dept. of Aero. and Astro. MIT, Draper Fellow.

Define $x_e(t)$ as the estimate of the state $x(t)$ computed by integrating the outputs of the redundant sensors with errors δb and δc, and with the initial condition in error by $\delta x(t_0)$. Let the state error $\delta x(t)$ be defined as

$$\delta x(t) = x(t) - x_e(t) \tag{3}$$

If the nonlinear dynamics of the state $x(t)$ are linearized about $x_e(t)$, then to first order the dynamics of the error $\delta x(t)$ take the linear form

$$\delta \dot{x}(t) = F(t)\delta x(t) + G(t)e \tag{4}$$

with initial condition $\delta x(t_0)$. $F(t)$ is the 9×9 dynamics matrix of the attitude, position, and velocity errors.[7] The G matrix couples the error sources into the state dynamics.[5] Partitioning G to be compatible with e yields

$$G(t) = [0 \,\vdots\, G_b(t) \,\vdots\, G_c(t) \,\vdots\, 0]$$

The recursive solution for Eq. (4) is

$$\delta x(t_n) = \Phi_x(t_n,t_{n-1})\delta x(t_{n-1}) + [0 \,\vdots\, \Phi_b(t_n,t_{n-1}) \,\vdots\, \Phi_c(t_n,t_{n-1}) \,\vdots\, 0]e \tag{5}$$

In Eq. (5), Φ_x is the 9×9 transition matrix associated with F. Φ_b relates the contribution of errors in the inertial sensor constant biases to state errors. Similarly, Φ_c couples the dynamic sensor errors δc into state errors.

This solution can also be expressed as a function of initial values as[8]

$$\delta x(t_n) = SX(t_n,t_0)\delta x(t_0) + [0 \,\vdots\, SB(t_n,t_0) \,\vdots\, SC(t_n,t_0) \,\vdots\, 0]e \tag{6a}$$

Recognizing that $\delta x(t_0)$ represents the first nine elements of e, Eq. (6a) can be rewritten as

$$\delta x(t_n) = S(t_n,t_0)e \tag{6b}$$

where the sensitivity matrix $S(t_n,t_0)$ is defined as

$$S(t_n,t_0) = [SX(t_n,t_0) \,\vdots\, SB(t_n,t_0) \,\vdots\, SC(t_n,t_0) \,\vdots\, 0] \tag{6c}$$

The recursive solution for $S(t_n,t_0)$ developed from Eqs. (5) and (6) is

$$S(t_n,t_0) = \Phi_x(t_n,t_{n-1})S(t_{n-1},t_0) + [0 \,\vdots\, \Phi_b(t_n,t_{n-1}) \,\vdots\, \Phi_c(t_n,t_{n-1}) \,\vdots\, 0] \tag{7a}$$

with initial condition

$$S(t_0,t_0) = [I \,\vdots\, 0 \,\vdots\, 0 \,\vdots\, 0] \tag{7b}$$

As Eq. (6) indicates, the navigation aid bias errors $\delta\gamma$ make no direct contribution to $\delta x(t)$. It is convenient, however, to include $\delta\gamma$ in the navigation system error vector e, since the errors $\delta\gamma$ affect the ability to improve knowledge of e [and consequently $x(t)$] through the use of navigation aids.

The sensitivity matrix of Eqs. (6) defines a linear relation between the error source vector e and the state error δx of Eq. (3). If the vector is assumed to be Gaussian with zero mean and covariance C_e, it follows that $\delta x(t_n)$ is also Gaussian, zero-mean with covariance

$$C_x(t_n) = S(t_n,t_0)C_e S^T(t_n,t_0)$$

In the absence of system failures, this result with Eqs. (6) and (7) gives the solution for the state error δx and its covariance in terms of e and its covariance.

The effects of failures in the elements of e are now considered. Each failure is modeled as a deterministic constant with three attributes: magnitude, duration, and onset time. Being deterministic, a failure does not affect the covariance of δx; however, it does affect its mean value. Consider a failure of magnitude b which occurs in sensor j at time t_f. Assume that the FDI system correctly detects and removes the failed sensor at time t_r. The presence of the failure causes the dynamics of the state error in Eq. (4) to be altered as

$$\delta \dot{x}(t) = F(t)\delta x(t) + G(t)[b(t_r,t_f)u_j + e] \tag{8}$$

where $b(t_r,t_f)$ is a step function of magnitude b starting at t_f and ending at t_r. u_j is an indicator vector identifying the failed sensor. All components of u_j are zero except the jth component which is unity. The solution to Eq. (8) is

$$\delta x(t_n) = SX(t_n,t_r)SB(t_r t_f)u_j b + S(t_n,t_0)e \tag{9}$$

where SX and SB are submatrices of the sensitivity matrix partitioned as

$$S(t_i,t_j) = [SX(t_i,t_j) \,\vdots\, SB(t_i,t_j) \,\vdots\, SC(t_i,t_j) \,\vdots\, S\gamma(t_i,t_j)] \tag{10}$$

In Eq. (9), if $t_r > t_n$ then SX and SB are evaluated at $t_r = t_n$. If $t_f > t_n$, then $SB = 0$. The expected value of the state error in Eq. (9) is (recall e is zero-mean)

$$\overline{\delta x}(t_n) = SX(t_n,t_r)SB(t_r,t_f)u_j b \tag{11}$$

Therefore, the mean error $\overline{\delta x}(t_n)$ is proportional to the magnitude of the sensor failure. Although the failure itself has no direct effect on the state error covariance C_x, a reconfiguration will affect the covariance of δx about its mean. The elimination of sensors due to reconfiguration decreases the measurement redundancy which causes an increase in the covariance of δx.

The previous discussion addresses the sensitivity of state errors to nominal sensor errors and sensor failures when no navigation aid information is available. The analysis is now extended to determine system performance with navigation aids. Again, both the absence and presence of system failures will be considered. Within the framework of the sensitivity analysis, performance improvements due to navigation aids translate into improving knowledge of the error source vector e.

It is assumed that all measurements can be linearized about $x_e(t)$ as

$$\delta z(t) = H_x(t)\delta x(t) + H_\gamma \delta\gamma + \nu(t) \tag{12}$$

where $\nu(t)$ is the random (white) measurement noise. Equation (12) can be rewritten as a function of the navigation error sources as

$$\delta z(t) = H_e(t)e + \nu(t) \tag{13}$$

where

$$H_e(t) = H_x(t)S(t,t_0) + [0 \,\vdots\, 0 \,\vdots\, 0 \,\vdots\, H_\gamma(t)] \tag{14}$$

Define the error in estimating e after the ith measurement $z(t_i)$ as (recall e is time invariant)

$$\tilde{e}_i = e - \hat{e}_i \tag{15}$$

Here the subscript i explicitly indicates the number of measurements which have contributed to the estimate $\hat{e}_i$. Given that $\hat{e}_0$ is zero, it follows that

$$\tilde{e}_0 = e \tag{16}$$

Therefore, the initial estimation-error covariance is equal to the error source covariance

$$C_{\tilde{e}_0} = E[\tilde{e}_0\tilde{e}_0^T] = E[ee^T] = C_e \tag{17}$$

If the measurements are processed in a minimum variance estimator, the estimation-error covariance after the *i*th measurement will be

$$C_{\tilde{e}_i} = [I - K_iH_e(t_i)]C_{\tilde{e}_{i-1}} \tag{18}$$

with initial condition given in Eq. (17).

In Eq. (18),

$$K_i = C_{\tilde{e}_{i-1}}H_e^T(t_i)[H_e(t_i)C_{\tilde{e}_{i-1}}H_e^T(t_i) + R_i]^{-1} \tag{19}$$

and

$$R_i = E[\nu(t_i)\nu(t_i)^T] \tag{20}$$

Given the estimate of e after i measurements $\hat{e}_i$, the estimate of the state error is

$$\widehat{\delta x}_i(t) = S(t,t_0)\hat{e}_i \tag{21}$$

and the error in the estimate of δx is

$$\widetilde{\delta x}_i(t) = S(t,t_0)\tilde{e}_i \tag{22}$$

To obtain an estimate of the navigation state $x(t)$ after i measurements, Eqs. (21) and (3) yield

$$\hat{x}_i(t) = \widehat{\delta x}_i(t) + x_e(t) \tag{23a}$$

or in terms of $\hat{e}_i$,

$$\hat{x}_i(t) = S(t,t_0)\hat{e}_i + x_e(t) \tag{23b}$$

The covariance of the error in the estimate of $x(t)$ given i measurements, $C_{\tilde{x}_i}(t)$, is the covariance of $\widetilde{\delta x}_i(t)$. From Eq. (22),

$$C_{\tilde{x}_i}(t) = E[\widetilde{\delta x}_i(t)\widetilde{\delta x}_i^T(t)] = S(t,t_0)C_{\tilde{e}_i}S^T(t,t_0) \tag{24}$$

In Eqs. (21-24), t can be greater than, equal to, or less than the time of the *i*th measurement t_i. If $t>t_i$, a predictor results. If $t<t_i$, then a smoothed solution is obtained.

A failure in one of the system error sources causes a bias in the estimates of e and δx. The relation between the estimation bias and the failure magnitude is derived below. First, consider the update equation for the estimate of the error source vector e. Using the first-order approximation for $z(t_i)$,

$$\hat{e}_i = \hat{e}_{i-1} + K_i[\delta z(t_i) - \widehat{\delta z}_{i-1}(t_i)] \tag{25}$$

where

$$\widehat{\delta z}_{i-1}(t_i) = H_e(t_i)\hat{e}_{i-1} \tag{26}$$

Substituting the solution for $\delta x(t)$ under failure conditions [Eq. (9)] into Eq. (12) gives

$$\delta z(t_i) = H_x(t_i)S(t_n,t_0)e + H_x(t_i)SX(t_i,t_r)SB(t_r,t_f)bu_j + H_\gamma(t_i)\delta\gamma + \nu(t_i) \tag{27}$$

Differencing Eqs. (26) and (27) and noting the definition of $H_e(t_i)$ in Eq. (14) results in

$$\delta z(t_i) - \widehat{\delta z}_{i-1}(t_i) = H_e(t_i)(e - \hat{e}_{i-1}) + H_x(t_i)SX(t_i,t_r)SB(t_r,t_f)bu_j + \nu(t_i) \tag{28}$$

Finally, substituting Eq. (28) into Eq. (25) and using the definition for the estimation error $\tilde{e}_i$ in Eq. (15) gives the following recursive form for the estimation error:

$$\tilde{e}_i = [I - K_iH_e(t_i)]\tilde{e}_{i-1} - K_iH_x(t_i)SX(t_i,t_r)SB(t_r,t_f)bu_j - K_i\nu(t_i) \tag{29}$$

with initial condition $\tilde{e}_0$ given. If $t_r > t_i$, then set $t_r = t_i$. If $t_f > t_i$, then $SB(t_r,t_f) = 0$.

Equation (29) can be solved in closed form as a function of the initial error $\tilde{e}_0$, the failure magnitude, and the measurement noise sequence. First, it will be convenient to write

$$\tilde{e}_i = A_i\tilde{e}_{i-1} + B_iu_jb - K_i\nu(t_i) \tag{30a}$$

where

$$A_i = I - K_iH_e(t_i) \tag{30b}$$

and

$$B_i = -K_iH_x(t_i)SX(t_i,t_r)SB(t_r,t_f) \tag{30c}$$

Equations (30) have the following solution

$$\tilde{e}_i = \Lambda(i,0)\tilde{e}_0 + \Gamma_iu_jb - \sum_{j=1}^{i}\Lambda(i,j)K_j\nu(t_j) \tag{31}$$

where

$$\Lambda(i,j) = \begin{cases} \prod_{k=1}^{i-j} A_{i-k} & (i>j) \\ I & (i=j) \end{cases} \tag{32}$$

and Γ_i satisfies the recursion

$$\Gamma_{i+1} = A_i\Gamma_i + B_i \qquad \Gamma_1 = B_1 \tag{33}$$

Recall that with no failures the mean value of the estimation-error $\tilde{e}_i$ is zero. Taking the expectation of Eq. (31) gives the mean value of $\tilde{e}_i$ under the failure mode as

$$\bar{\tilde{e}}_i = \Lambda(i,0)\bar{\tilde{e}}_0 + \Gamma_iu_jb \tag{34}$$

where $\bar{\tilde{e}}_0$ is explicitly included in order to model calibration biases which might also be categorized as failures. The covariance of $\tilde{e}_i$ about its mean $\bar{\tilde{e}}_i$ is also found from Eq. (31) to be

$$C_{\tilde{e}_i} = E[(\tilde{e}_i - \bar{\tilde{e}}_i)(\tilde{e}_i - \bar{\tilde{e}}_i)^T] = \Lambda(i,0)C_{\tilde{e}_0}\Lambda(i,0)^T + \mathcal{R}_i \tag{35}$$

where

$$\mathcal{R}_i = \sum_{j=1}^{i}\Lambda(i,j)K_jR_jK_j^T\Lambda^T(i,j) \tag{36}$$

If we define

$$\hat{S}_i(t,0) = S(t,0)\Lambda(i,0) \tag{37}$$

then the mean and variance of the estimation error for δx are

$$\overline{\widetilde{\delta x}}_i(t) = \hat{S}_i(t,0)\bar{\tilde{e}}_0 + S(t,0)\Gamma_iu_jb \tag{38}$$

and

$$C_{\tilde{x}_i} = \hat{S}_i(t,0)C_{\tilde{e}_0}\hat{S}_i^T(t,0) + S(t,0)\mathcal{R}_iS^T(t,0) \tag{39}$$

$\hat{S}_i(t,0)$ in Eq. (37) will be referred to as the *updated* sensitivity matrix, since it relates the contributions of the mean and variance of the initial estimation error of e to the mean and variance of the estimation error of δx through Eqs. (38) and (39).

Several observations can be made concerning the implementation of this linearized analysis. Consider the usual situation in which the statistics of $\delta x(t)$ are of interest and there are no failures in the system. The sensitivity matrix $S(t,t_0)$ is used to compute the covariance $C_x(t)$ of $\delta x(t)$ for any convariance C_e of the vector e of error sources. The same $S(t,t_0)$ is used to compute the covariance $C_{\tilde{x}}(t)$ of $\widetilde{\delta x}(t)$ when the covariance of $\tilde{e}$, $C_{\tilde{e}}$ has been updated according to a particular measurement schedule. To compute $C_{\tilde{e}}$ at measurement time t_i, $S(t_i, t_0)$ is required. It is thus possible to study the effects of various values of (and correlations among) the errors and various update schedules by precomputing (only once) and storing $S(t,t_0)$ for t_r and all t_i of interest, and computing $C_{\tilde{e}}$ separately for each measurement schedule. In failure cases, these same efficiencies can be realized, provided that the prestored data is sufficient to recover the updated sensitivity $\tilde{S}(t,t_0)$ and the bias sensitivity Γ_i. The recovery of Γ_i, for example, requires that the time histories of the matrices used to propagate $S(t,t_0)$ in Eq. (7a) also be stored in order to construct the appropriate $SX(t_i,t_r)$ and $SB(t_r,t_f)$ matrices.

The aided navigation system performance, which is predicted from the analysis described in this paper, assumes that a large number of states are estimated at each measurement time. In practical applications, where only a limited number of states are estimated, this predicted performance will be degraded. In order to quantify this degradation, the sensitivity analysis can be reformulated in a manner which is analogous to a suboptimal covariance analysis.[4]

Error covariances represent the primary results of most linearized error analyses of navigation system performance. In addition to these covariances, the linearized analysis described in this paper provides the sensitivities of the state errors to each of the navigation system error sources and to any failure which can be modeled as a bias in the system. These sensitivities provide considerable insight into the performance of the aided redundant navigation system in the normal mode (i.e., in the absence of failures), as well as in any of the operational modes resulting from failures or FDI decisions which affect performance. However, for making comparisons among systems with different characteristics (i.e., different initial conditions, sensor errors, failure modes, FDI structures, update schedules, navaid errors, etc.), the means, covariances, and sensitivities are sometimes inconvenient. For these comparisons, a scalar figure-of-merit (FOM) is preferred. A FOM which is indicative of the probability of mission failure can be obtained from the mean and covariance of $\widetilde{\delta x}$.

Given the redundant navigation system and the mission profile, the probability of experiencing a particular history of failures at times $\{t_f\}$ and system reconfigurations at times $\{t_r\}$ can be determined from a Markov model of the system.[6] Let the probability of the kth history be P_k, and let the state error for the kth history be $\widetilde{\delta x}_k$. If all error sources are Gaussian, the mean and covariance of $\widetilde{\delta x}_k$, which are obtained via the sensitivity analysis, completely specify the probability density function $p_k(\widetilde{\delta x})$. If the system performance requirement is expressed in terms of bounds on $\widetilde{\delta x}$, the probability of exceeding these bounds can be determined (at least conceptually) by integrating $p_k(\widetilde{\delta x})$ over the appropriate region in the domain of $\widetilde{\delta x}$. It is often possible to identify two critical components $\widetilde{\delta x}_c$ of $\widetilde{\delta x}$ which have significantly higher probabilities of exceeding their specified bounds. In these cases, the figure-of-merit FOM_k for the kth history is approximately

$$\text{FOM}_k \approx 1 - \int\int_{S_c} p_k(\widetilde{\delta x}_c)\,\widetilde{\delta x}_c \tag{40}$$

where S_c represents the bounds on the two components of δx_c. The integral can be evaluated using a series approximation.[9] The expected value of FOM_k over the ensemble of possible histories is

$$\text{FOM} = \sum_k \text{FOM}_k P_k \tag{41}$$

If the system performance requirements are expressed as bounds on the errors $\delta\tilde{y}$ in a set of orbital elements y, the statistics of the orbital elements' variations can be related to the statistics of $\delta\tilde{x}$ by linearizing the relation[10]

$$y(t) = a(x(t)) \tag{42}$$

to obtain a matrix A such that[5]

$$\widetilde{\delta y}(t) = A(t)\widetilde{\delta x}(t) \tag{43}$$

The statistics of $\delta\tilde{y}$ can then be obtained from the statistics of $\delta\tilde{x}$. The FOM_k of Eq. (40) is evaluated in terms of the bounds on y and the probability density $p_k(\delta\tilde{y})$.

Results

To illustrate the application of the results of the previous section, the performance of a strapdown inertial navigation system is evaluated for a multistage booster used to insert a payload into geosynchronous orbit. The trajectory phases and their approximate durations are indicated in Table 1.

The redundant strapdown inertial navigation system is comprised of five gyros and five accelerometers in a conical configuration. By observing sensor outputs, it is possible to detect and identify a single sensor failure for this configuration.[11]

The standard deviations of the navigation system error sources are presented in Table 2. These values are representative of inertial-grade sensors, but are not intended to characterize any particular manufacturer's sensors.

The linearized analysis described in the previous section is used to evaluate system performance for the redundant inertial navigation system in the normal mode (i.e., in the absence of failures) and in various failure and reconfiguration modes. The results for the latter modes provide insight into navigation system performance in the presence of sensor failures and in situations in which one or more sensors have been removed from the system due to FDI decisions. These FDI decisions may include false alarms and incorrect identifications as well as correct identifications of sensor failures.

The linearized analysis is also used to determine the extent to which navigation system performance is improved through the use of navigation aids. Attitude and position updates are considered. The attitude updates are made with a slit-type star sensor[5] consisting of four slits. The standard deviation of the uncertainty in the alignment between the star sensor and the nav-base is assumed to be 42 s per axis. The standard deviation of the measurement noise is assumed to be 10 arc sec. All star-sensor measurements are performed during the attitude maneuver immediately preceding the insertion burn. Two stars separated by approximately 60 deg are simulated.

Table 1 Trajectory phases

Phase	Duration, s
Launch	500
Low-Earth orbit	3505
Attitude maneuver	130
Transfer burn	120
Transfer orbit	16650
Attitude maneuver	320
Insertion burn	65

Table 2 Standard deviation of navigation system error sources

Error sources		Standard deviation
Initial conditions		
Alignment		
Azimuth		50 arc sec
Level		4 arc sec
Position		0.0
Velocity		0.0
Gyro error sources		
Bias drift		0.010 deg/h
Scale factor		50 ppm
Misalignment about OA		15 arc sec
Misalignment about SA		15 arc sec
g-sensitive drift	IA	0.050 deg/h/g
	SA	0.075 deg/h/g
	OA	0.020 deg/h/g
g^2-sensitive drift	IA,SA	0.020 deg/h/g^2
	IA,OA	0.001 deg/h/g^2
	IA,IA	0.001 deg/h/g^2
	OA,SA	0.015 deg/h/g^2
	SA,SA	0.003 deg/h/g^2
Accelerometer error sources		
Bias drift		40 μg
Scale factor		50 ppm
Misalignment about OA		25 arc sec
Misalignment about PA		25 arc sec

Table 3 Standard deviations of $\widetilde{\delta x}$ at insertion for normal mode

Component	Without update	With attitude update	With attitude and position updates
Quaternion, μrad			
X	313.60	63.60	60.60
Y	341.00	56.30	56.20
Z	402.00	44.50	44.00
Position, n. mi.			
T	20.68	20.47	0.19
N	12.57	12.47	0.19
R	16.17	16.12	0.19
Velocity, fps			
$\dot{T}$	3.75	3.45	0.34
$\dot{N}$	4.07	2.29	0.67
$\dot{R}$	16.69	15.81	0.73

Four measurements (one per slit) are made on each star as the star-sensor line-of-sight rotates past the star.

Ranging measurements are added primarily for purposes of comparison. The ranging device is left unspecified, but is assumed to have an accuracy of 500 m (1 σ). Position updates are accomplished by interleaving three orthogonal range measurements among the four slit transit measurements of each star for a total of six range measurements.

Tables 3-6 present the statistics of the vehicle state errors $\widetilde{\delta x}$ at insertion into geosynchronous orbit for the normal mode and for particular failure modes. Position and velocity errors at insertion are presented in a tangential-normal-radial (TNR) frame because system performance specifications are frequently stated in this frame. The radial axis is along $-r$ and the normal axis is along $v \times r$, where r and v are the Earth-centered inertial (ECI) position and velocity vectors.

Table 3 presents the standard deviations of the attitude, position, and velocity errors at insertion into geosynchronous orbit for the normal mode. The results summarized in Table 3 also indicate the extent to which navigation system performance is improved by the attitude and position measurements described above. Since attitude errors couple into velocity errors during burns, an attitude update prior to the insertion burn reduces the velocity errors at insertion.

Table 4 Means of $\widetilde{\delta x}$ at insertion for 0.1 deg/h undetected gyro failure present throughout mission

Component	Without update	With attitude update	With attitude and position updates
Quaternion, μrad			
X	214.40	−2.00	3.00
Y	−2201.50	−63.20	−59.30
Z	690.40	40.00	32.40
Position, n. mi.			
T	8.13	−5.66	−0.01
N	3.71	14.07	0.02
R	9.86	1.55	−0.00
Velocity, fps			
$\dot{T}$	0.91	−0.12	−0.09
$\dot{N}$	3.24	1.21	−0.02
$\dot{R}$	25.71	1.87	0.05

Table 5 Means of $\widetilde{\delta x}$ at insertion for 0.1 deg/h gyro failure which occurs in transfer burn and remains undetected

Component	Without update	With attitude update
Quaternion, μrad		
X	−355.80	−25.00
Y	−2264.90	−51.10
Z	723.10	52.50
Position, n. mi.		
T	−0.03	−13.94
N	0.03	10.16
R	0.05	−7.96
Velocity, fps		
$\dot{T}$	0.59	−2.87
$\dot{N}$	9.72	1.64
$\dot{R}$	18.15	−5.99

Table 6 Means of $\widetilde{\delta x}$ at insertion for 400 μg undetected accelerometer failure present throughout mission

Component	Without update	With attitude update	With attitude and position updates
Quaternion, μrad			
X	0.00	0.00	0.10
Y	0.00	0.00	0.50
Z	0.00	0.00	1.80
Position, n. mi.			
T	−39.85	−39.85	−0.01
N	−21.95	−21.95	−0.00
R	−31.15	−31.15	0.01
Velocity, fps			
$\dot{T}$	− 6.83	− 6.83	−0.21
$\dot{N}$	− 3.89	− 3.89	0.26
$\dot{R}$	−30.60	−30.60	0.08

Although the sensitivities of velocity errors to gyro bias drifts are not presented here, examining these sensitivities indicates that the attitude update effectively removes the gyro bias drifts' contributions to the unaided velocity errors at insertion. Since an attitude update yields no direct information about position, it does not result in any significant reduction of position errors. When ranging measurements are interleaved with the star-sensor measurements, on the other hand, both position and velocity errors are substantially reduced.

System performance for three arbitrarily selected failure modes is presented in Tables 4-6. The presence of bias failures

Table 7 Navigation accuracy requirements

Position, n. mi.	
T	45
N	30
R	30
Velocity, fps	
T	8
N	8
R	40

Table 8 Navigation accuracy figures-of-merit

Case	Without update	With attitude update	With attitude and position updates
Normal mode	0.1016	0.0640	$<10^{-6}$
0.1 deg/h undetected gyro failure throughout	0.2608	0.1285	$<10^{-6}$
400 μg undetected accelerometer failure throughout	0.5562	0.5307	$<10^{-6}$

does not affect the covariance of the navigation errors as long as no reconfiguration of the system occurs. The mean estimation errors for both the system error sources $\boldsymbol{e}$ and the state error $\delta x(t)$, however, may become nonzero when a failure occurs [see Eqs. (34) and (38)]. Since the filter has no knowledge of failure onset time, the biased estimate of $\boldsymbol{e}$ is presumed to be present in the system from t_0 [see Eq. (21)]. A failure, of course, may occur at any time during the mission.

The means of the elements of δx caused by a gyro failure of 0.1 deg/h, which remains undetected throughout the mission, are listed in Table 4. When attitude updates are performed, many of the mean values of the estimation errors are large for those elements of $\boldsymbol{e}$ which are associated with attitude (bias drift, initial attitude errors, etc.). This reflects the inability of the filter to assign the gyro bias failure to its single source within $\boldsymbol{e}$. As a result, although the means of the attitude and velocity components of δx are substantially reduced, the mean of the normal position error increases (see Table 4). This increase in the means of the position components of δx was observed to be even more pronounced in cases in which the failure onset time came later in the mission (see Table 5).

The error means caused by a 400 μg accelerometer failure which remains undetected throughout the flight are presented in Table 6. The attitude update has no effect on the means in this case. The ranging measurements are quite effective in reducing the means of the velocity and position errors, at the cost of a slight increase in attitude means. As in the gyro case, this increase is due to the inability of the filter to assign the bias failure to the correct, single element of $\boldsymbol{e}$.

The results from Tables 3-6 may be interpreted in terms of the FOM_k given by Eq. (40). In order to do so, performance bounds for the navigation errors at insertion are assumed. These requirements, shown in Table 7, are representative of achievable performance for insertion into a geosynchronous orbit.

In addition to the data in Table 3, knowledge of the correlations among the navigation errors is required to evaluate the two-dimensional Gaussian probability density function given in Eq. (40). These correlations are provided by the linearized analysis. The FOM_k for the nine cases presented in Tables 3, 4, and 6 are summarized in Table 8. For each case, this FOM_k may be interpreted as the probability of exceeding the navigation accuracy bounds of Table 7.

As indicated in Table 8, the FOM_k are reduced for both normal and failure cases when navigation aid measurements are incorporated. These reduced FOM_k reflect a decreased sensitivity to particular system error sources as well as, in the failure cases, the effects of sensor failures. The use of updates thus affords an opportunity to relax the FDI thresholds which would otherwise be required in the absence of updates. A Markov model of the redundant system and its associated redundancy management structure can be used[12] to optimize the choice of these thresholds for the FOM given in Eq. (41). This FOM is a function of the costs FOM_k associated with each of the operational state time histories and the probabilities P_k with which these time histories occur. The updates serve to reduce the costs FOM_k, due to the reduced sensitivities to error sources and sensor failure effects. Relaxing the thresholds may either decrease (as in the case of time histories involving false alarms) or increase (as in the case of time histories involving undetected failures) the probabilities P_k. The resulting FOM depends on how these effects combine. In general, a net improvement in the FOM is expected, since the effects of those P_k which increase are not likely to offset the effects of those P_k and the FOM_k which are reduced.

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